A REVIEW OF COMPLEX NUMBERS

Q: WHAT IS $\sqrt{-1}$?

A: THERE IS NO ANSWER IN THE SET OF REAL NUMBERS, $\mathbb{R}$. TO ALLOW FOR AN ANSWER, THE IMAGINARY NUMBER j IS INVENTED SO THAT $j^2 = -1$, THUS $\sqrt{-1} = \pm j$

(MATHEMATICIANS USE $\sqrt{-1} = i$, BUT WE USE j SINCE i = CURRENT)

WE DEFINE THE SET OF COMPLEX NUMBERS, $\mathbb{C}$, AS AN ORDERED PAIR OF REAL NUMBERS, $a + jb$ WHERE $a, b \in \mathbb{R}$

$$\mathrm{Re}\{a+jb\} \triangleq a$$

$$\mathrm{Im}\{a+jb\} \triangleq b$$

EXAMPLE: Q: WHAT ARE THE SOLUTIONS OF $x^2 + 4x + 13 = 0$

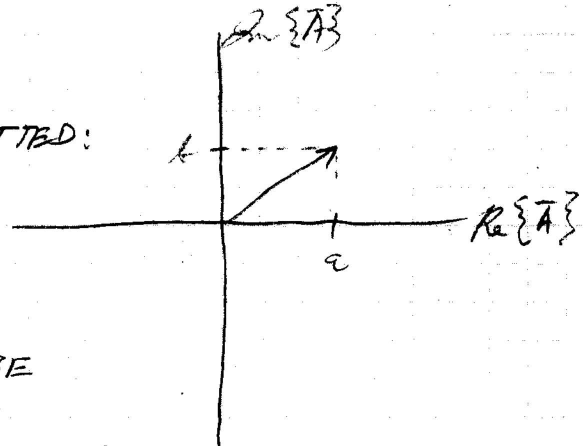
A: APPLY QUADRATIC FORMULA: $x = \dfrac{-4 \pm \sqrt{4^2 - 4(13)}}{2} = -2 \pm 3j$

NOTATION: LET BOLDFACE (OVERBAR) REPRESENT A COMPLEX QUANTITY

$$\bar{x} = -2 \pm 3j$$

THE COMPLEX PLANE

LET $\bar{A} = a + jb$. THIS CAN BE PLOTTED: AS A VECTOR



THE NOTATION $\bar{A} = a + jb$ IS CALLED RECTANGULAR NOTATION

THE COMPLEX NUMBER $\bar{A}$ CAN ALSO BE DESCRIBED BY MAGNITUDE AND ANGLE

LET $c = \sqrt{a^2 + b^2}$ AND $\theta = \arg(\bar{A})$

WHERE
$$\arg(a+jb) = \begin{cases} \tan^{-1}\frac{b}{a} & \text{IF } a > 0 \\ \tan^{-1}\frac{b}{a} - \pi \text{ radians} & \text{IF } a < 0 \text{ AND } b < 0 \\ \tan^{-1}\frac{b}{a} + \pi \text{ radians} & \text{IF } a < 0 \text{ AND } b \geq 0 \\ \frac{\pi}{2} & \text{IF } a = 0 \text{ AND } b > 0 \\ -\frac{\pi}{2} & \text{IF } a = 0 \text{ AND } b < 0 \\ \text{UNDEFINED} & \text{IF } a = b = 0 \end{cases}$$

THEN WE SAY $\bar{A} = c\angle\theta$

CALLED POLAR NOTATION

SOMETIMES THE ANGLE IS GIVEN IN DEGREES, $360° = 2\pi$ radians

Q: WHEN ARE TWO COMPLEX NUMBERS EQUAL?

A: SUPPOSE $\bar{A} = a + jb = c\angle\theta$ AND $B = x + jy = z\angle\phi$

$\bar{A} = \bar{B}$ IFF $(a = x$ AND $b = y)$ OR $(c = z$ AND $\theta = \phi)$ OR $(c = -z$ AND $\theta = \phi \pm \pi$ rad$)$

WATCH FOR THIS!

EULER'S FORMULA — DERIVED FROM TAYLOR SERIES $\to e^{j\theta} = \cos\theta + j\sin\theta$

TAYLOR: $f(x) = \sum_{n=0}^{\infty} \left.\frac{d^n f(x)}{dx^n}\right|_{x=a} \cdot \frac{(x-a)^n}{n!}$ ←

Auguries of Innocence
To see a world in a grain of sand
And a heaven in a wild flower
Hold infinity in the palm of your hand
And eternity in an hour
— William Blake

ie TAYLOR SERIES EXPANSION OF $e^{j\theta}$ AT $a=0$

$e^{j0} = 1$

$\left.\frac{de^{j\theta}}{d\theta}\right|_{\theta=0} = \left.je^{j\theta}\right|_{\theta=0} = j$

$\left.\frac{d^2e^{j\theta}}{d\theta}\right|_{\theta=0} = \left.j^2e^{j\theta}\right|_{\theta=0} = -1$

⋮

etc

$\therefore e^{j\theta} = 1 + j\theta + \frac{j^2}{2!}\theta^2 + \frac{j^3}{3!}\theta^3 + \frac{j^4}{4!}\theta^4 + \frac{j^5}{5!}\theta^5$

$= 1 + j\theta - \frac{1}{2!}\theta^2 - \frac{j}{3!}\theta^3 + \frac{1}{4!}\theta^4 + \frac{j}{5!}\theta^5 + \cdots$

NOW OBSERVE TAYLOR EXPANSION OF $\cos\theta$ AT $a=0$

$\left.\cos\theta\right|_{\theta=0} = 1$

$\left.\frac{d}{dt}\cos\theta\right|_{\theta=0} = \left.-\sin\theta\right|_{\theta=0} = 0$

$\left.\frac{d^2}{dt^2}\cos t\right|_{\theta=0} = \left.-\cos\theta\right|_{\theta=0} = -1$

⋮

etc

$\therefore \cos\theta = 1 + 0\theta - \frac{\theta^2}{2!} + 0\theta^3 + \frac{\theta^4}{4!} + 0\theta^5 - \frac{\theta^6}{6!} + \cdots$

AND TAYLOR EXPANSION OF $\sin\theta$ AT $a=0$

$\left.\sin\theta\right|_{\theta=0} = 0$

$\left.\frac{d}{d\theta}\sin\theta\right|_{\theta=0} = \left.\cos\theta\right|_{\theta=0} = 1$

$\left.\frac{d^2}{d\theta^2}\sin\theta\right|_{\theta=0} = \left.-\sin\theta\right|_{\theta=0} = 0$

$\left.\frac{d^3}{d\theta^3}\sin\theta\right|_{\theta=0} = \left.-\cos\theta\right|_{\theta=0} = -1$

$\therefore \sin\theta = 0 + \frac{\theta}{1!} + 0\theta^2 - \frac{\theta^3}{3!} + 0\theta^4 + \frac{\theta^5}{5!} + \cdots$

$\therefore j\sin\theta = 0 + jx + 0 - \frac{j\theta^3}{3!} + 0 + j\frac{\theta^5}{5!} + \cdots$

$$\boxed{\therefore e^{j\theta} = \cos\theta + j\sin\theta}$$

DORDT COLLEGE
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CORRILARIES OF EULER'S FORMULA

$$e^{-j\theta} = \cos\theta - j\sin\theta$$

$$\cos\theta = \frac{e^{j\theta}+e^{-j\theta}}{2}$$

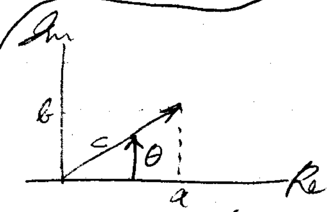
$$\sin\theta = \frac{e^{j\theta}-e^{-j\theta}}{2j}$$

NOTATION USED ON THIS PAGE;

AN OVERBAR OR OVER-ARROW AS AN "$\bar{A}$" OR "$\vec{A}$" STANDS FOR A REMINDER THAT THIS QUANTITY IS COMPLEX

e.g. $\vec{A} = a + bi$ WHERE $a, b \in \mathbb{R}$

BUT MATHEMATICIANS USE $\bar{A}$ TO MEAN COMPLEX CONJUGATE



APPLICATION OF EULER'S FORMULA:

IF $\bar{A} = a + jb = c\angle\theta$ THEN $\bar{A} = ce^{j\theta}$

OR IN OTHER WORDS $\angle\theta$ IS SHORTHAND FOR $e^{j\theta}$

PROOF: $ce^{j\theta} = c(\cos\theta + j\sin\theta) = c\cos\theta + jc\sin\theta = a + jb$

Q: WHAT IS $\bar{A}^2$? $\bar{A}^2 = (ce^{j\theta})^2 = c^2e^{j2\theta} = c^2\angle 2\theta$

Q: IF $\bar{B} = x + jy = z\angle\phi = ze^{j\phi}$ WHAT IS $\bar{A}\bar{B}$?

$$\bar{A}\bar{B} = (ce^{j\theta})(ze^{j\phi}) = cz\,e^{j(\theta+\phi)} = cz\angle\theta+\phi$$

ARITHMETIC OF COMPLEX NUMBERS

LET $\vec{A} = a + jb = c\angle\theta$ AND $\bar{B} = x + jy = z\angle\phi$

ADDITION (SUBTRACTION) — DO IT IN RECTANGULAR FORM

$$\bar{A} \pm \bar{B} = (a \pm x) + j(b \pm y)$$

MULTIPLICATION (DIVISION) — DO IT IN POLAR FORM

$$\bar{A}\bar{B} = cz\angle\theta+\theta$$

$$\frac{\bar{A}}{\bar{B}} = \frac{c}{z}\angle\theta-\phi$$

$3^3 = 27$

$\left(3\angle\frac{2\pi}{3}\right)^3 = 27$

$3\left(\frac{-2\pi}{3}\right)^3 = 27$

$\therefore \sqrt[3]{27} = 3\angle 0°$ OR $3\angle\frac{2\pi}{3}$ OR $3\angle\frac{-2\pi}{3}$

COMPLEX CONJUGATE IF $\bar{A} = a + jb$ THEN $\bar{A}^* = a - jb$

IF $\bar{A} = c\angle\theta$ THEN $\bar{A}^* = c\angle-\theta$

ROOTS OF COMPLEX NUMBERS

THERE ARE n nTH ROOTS OF A NUMBER

EXAMPLES: $(2j)^2 = -4 \quad \therefore \sqrt{-4} = 2j$

$(-2j)^2 = -4 \quad \therefore \sqrt{-4} = -2j$

$(1+j)^4 = \left(\sqrt{2}\angle\frac{\pi}{4}\right)^4 = -4$

$(1-j)^4 = \left(\sqrt{2}\angle\frac{-\pi}{4}\right)^4 = -4$

$(-1+j)^4 = \left(\sqrt{2}\angle\frac{3\pi}{4}\right)^4 = -4$

$(-1-j)^4 = \left(\sqrt{2}\angle\frac{-3\pi}{4}\right)^4 = -4$

$\Rightarrow \therefore \sqrt[4]{-4} = \pm 1 \pm j$